

**$(O(V \oplus F), O(V))$ IS A GELFAND PAIR
FOR ANY QUADRATIC SPACE V OVER A LOCAL FIELD F**

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ABSTRACT. Let V be a quadratic space with a form q over an arbitrary local field F of characteristic different from 2. Let $W = V \oplus Fe$ with the form Q extending q with $Q(e) = 1$. Consider the standard embedding $O(V) \hookrightarrow O(W)$ and the two-sided action of $O(V) \times O(V)$ on $O(W)$.

In this note we show that any $O(V) \times O(V)$ -invariant distribution on $O(W)$ is invariant with respect to transposition. This result was earlier proven in a bit different form in [vD] for $F = \mathbb{R}$, in [AvD] for $F = \mathbb{C}$ and in [BvD] for p -adic fields. Here we give a different proof.

Using results from [AGS], we show that this result on invariant distributions implies that the pair $(O(V), O(W))$ is a Gelfand pair. In the archimedean setting this means that for any irreducible admissible smooth Fréchet representation (π, E) of $O(W)$ we have $\dim \text{Hom}_{O(V)}(E, \mathbb{C}) \leq 1$.

A stronger result for p -adic fields is obtained in [AGRS07].

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1. INTRODUCTION

Let F be a local field of characteristic different from 2.

Let (W, Q) be a quadratic space defined over F and fix $e \in W$ a unit vector. Consider the quadratic space $V = e^\perp$ with $q = Q|_V$. Define the standard imbedding $O(V) \hookrightarrow O(W)$ and consider the two-sided action of $O(V) \times O(V)$ on $O(W)$ defined by $(g_1, g_2)h := g_1 h g_2^{-1}$. We also consider the anti-involution τ of O_Q given by $\tau(g) = g^{-1}$. In this paper we prove the following theorem

Key words and phrases. Multiplicity one, invariant distribution, orthogonal groups, Gelfand pairs.

2000 Mathematics Subject Classification Classification: 22E45, 20G05, 20G25, 46F99.

Theorem (A). *Any $O(V) \times O(V)$ invariant distribution on $O(W)$ is invariant under τ .*

This theorem has the following corollary in representation theory.

Theorem (B). *Let (π, E) be an irreducible admissible representation of $O(W)$. Then*

$$\dim \text{Hom}_{O(V)}(E, \mathbb{C}) \leq 1$$

Here admissible representation refers to the usual notion in the non-archimedean case and to the notion of admissible smooth Fréchet representation in the archimedean setting.

Our proof for the archimedean and non-archimedean case is uniform, except at one point where the archimedean case requires an extra analysis of a certain normal bundle (see lemma 4.2).

Remark 1.1. *We note that a related result for unitary representations of $SO(V, Q)$ is proved in [BvD] (for p -adic fields) and in [vD] (for the real numbers). In fact, the proof given in those papers implies also theorem A. Also, an analogous theorem for unitary groups is proven in [vD2].*

Acknowledgements. We thank Prof. Gerrit van Dijk for pointing out to us that the arguments of [vD], [AvD] and [BvD] give a proof of theorem A. We also thank Dr. Sun Binyong for finding a mistake in the previous version of this note. Finally, we would like to thank the referee for useful remarks.

2. FROM INVARIANT DISTRIBUTIONS TO REPRESENTATION THEORY

In this section we recall a technique due to Gelfand and Kazhdan which allows to deduce theorem B from theorem A.

Recall the following theorem ([AGS])

Theorem 2.1. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(T) = T$ for all bi H -invariant distributions¹ on G . Then for any irreducible admissible representation (π, E) of G we have*

$$\dim \text{Hom}_H(E, \mathbb{C}) \cdot \dim \text{Hom}_H(\tilde{E}, \mathbb{C}) \leq 1,$$

where \tilde{E} denotes the smooth contragredient representation.

Note that in the non-archimedean case the same result is proven in [Pra].

To finish the deduction of theorem B from theorem A we will show that

Theorem 2.2. *Let (π, E) be an irreducible admissible representation of $G = O(V)$. Then $\tilde{E} \cong E$ and in particular*

$$\dim \text{Hom}_H(E, \mathbb{C}) = \dim \text{Hom}_H(\tilde{E}, \mathbb{C})$$

For the proof we recall proposition I.2 (chapter 4) from [MVW]:

Proposition 2.3. *Let V be a quadratic space and let $g \in O(V)$. Then g is conjugate to g^{-1} .*

¹In fact it is enough to check this only for Schwartz distributions.

Proof of Theorem 2.2. For non-archimedean fields this is a theorem from [MVW] page 91. For archimedean fields we use the Harish-Chandra regularity theorem and the proposition that any element in $g \in O(V)$ is conjugate in $O(V)$ to g^{-1} . Thus, the characters of E and \tilde{E} are the same and hence $\tilde{E} \cong E$. \square

Remark 2.4. A related result for the groups $SO(V)$ can be found in [GP], proposition 5.3.

3. BASIC RESULTS ON INVARIANT DISTRIBUTIONS

In this paper we consider distributions over l -spaces and over smooth manifolds. l -spaces are locally compact totally disconnected topological spaces (see [BZ], section 1).

For X a smooth manifold or an l -space we denote by $\mathcal{D}(X)$ the space of distributions on X . When X is an l -space this means that $\mathcal{D}(X) = S(X)^*$ where $S(X)$ is the space of locally constant functions with compact support on X . For smooth X , we let $\mathcal{D}(X) = C_c^\infty(X)^*$.

The basic tools to study invariant distributions on a G -space X are Bruhat filtration, Frobenius reciprocity ([BZ], [Bar] and [AGS]) and the Bernstein's localization principle ([Ber] and [AG]). Let us remind the statements.

For the simplicity of formulation we provide, for each principle, two versions: for l -spaces and for smooth manifolds.

3.1. Bruhat Filtration. Although we will not need the non-archimedean version of this principle, we formulate it for completeness. It is a simple consequence of proposition 1.8 in [BZ].

Theorem 3.1. *Let an l -group G act on an l -space X . Let $X = \bigcup_{i=0}^l X_i$ be a G -invariant stratification of X . Let χ be a character of G . Suppose that $\mathcal{D}(X_i)^{G, \chi} = 0$. Then $\mathcal{D}(X)^{G, \chi} = 0$.*

To formulate the archimedean version we let X be a smooth manifold and $Y \subset X$ a smooth submanifold. We remind the definition of the conormal bundle CN_Y^X . For this denote by T_X the tangent bundle of X and by $N_Y^X := (T_X|_Y)/T_Y$ the normal bundle to Y in X . The conormal bundle is defined by $CN_Y^X := (N_Y^X)^*$. Denote by $Sym^k(CN_Y^X)$ the k -th symmetric power of the conormal bundle.

Theorem 3.2. *Let a real reductive group G act on a smooth affine real algebraic variety X . Let $X = \bigcup_{i=0}^l X_i$ be a smooth G -invariant stratification of X . Let χ be an algebraic character of G . Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and any $0 \leq i \leq l$ we have $\mathcal{D}(X_i, Sym^k(CN_{X_i}^X))^{G, \chi} = 0$. Then $\mathcal{D}(X)^{G, \chi} = 0$.*

For proof see [AGS], section B.2.

3.2. Frobenius reciprocity. For l -space, the following version of Frobenius reciprocity is proven in [Ber]:

Theorem 3.3 (Frobenius reciprocity). *Let a unimodular l -group G act transitively on an l -space Z . Let $\varphi : X \rightarrow Z$ be a G -equivariant continuous map. Let $z \in Z$. Suppose that its stabilizer $\text{Stab}_G(z)$ is unimodular. Let X_z be the fiber of z . Let χ be a character of G . Then $\mathcal{D}(X)^{G, \chi}$ is canonically isomorphic to $\mathcal{D}(X_z)^{\text{Stab}_G(z), \chi}$.*

An archimedean version is considered in [Bar]. Here is a slight generalization (see [AGS]):

Theorem 3.4 (Frobenius reciprocity). *Let a unimodular Lie group G act transitively on a smooth manifold Z . Let $\varphi : X \rightarrow Z$ be a G -equivariant smooth map. Let $z \in Z$. Suppose that its stabilizer $\text{Stab}_G(z)$ is unimodular. Let X_z be the fiber of z . Let χ be a character of G . Then $\mathcal{D}(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_z)^{\text{Stab}_G(z),\chi}$. Moreover, for any G -equivariant bundle E on X , $\mathcal{D}(X, E)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_z, E|_{X_z})^{\text{Stab}_G(z),\chi}$.*

3.3. Bernstein's Localization principle. For l -spaces it is taken from [Ber]:

Theorem 3.5 (Localization principle). *Let X and T be l -spaces and $\phi : X \rightarrow T$ be a continuous map. Let an l -group G act on X preserving the fibers of ϕ . Let χ be a character of G . Suppose that for any $t \in T$, $\mathcal{D}(\phi^{-1}(t))^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.*

For real smooth algebraic varieties, the following theorem is proven in [AG], Corollary A.0.3:

Theorem 3.6 (Localization principle). *Let a real reductive group G act on a smooth affine real algebraic variety X . Let Y be a smooth real algebraic variety and $\phi : X \rightarrow Y$ be an algebraic G -invariant submersion. Suppose that for any $y \in Y$ we have $\mathcal{D}(\phi^{-1}(y))^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.*

4. PROOF OF THEOREM A

Recall the setting. (W, Q) is a quadratic space over F , $e \in W$ with $Q(e) = 1$. Also (V, q) is defined by $V = e^\perp$ and $q = Q|_V$.

We need some further notations.

- $O_q = O(V, q)$ is the group of isometries of the quadratic space (V, q) .
- $G_q = O(V, q) \times O(V, q)$.
- $\Delta : O_q \rightarrow G_q$ the diagonal. $H_q = \Delta(O_q) \subset G_q$.
- $\tau(g_1, g_2) = (g_2, g_1)$.
- $\widetilde{G}_q = G_q \rtimes \{1, \tau\}$, same for \widetilde{H}_q .
- $\chi : \widetilde{G}_q \rightarrow \{+1, -1\}$ the non trivial character with $\chi(G_q) = 1$.
- \widetilde{G}_Q acts on O_Q by $(g_1, g_2)x = g_1 x g_2^{-1}$ and $\tau(x) = x^{-1}$.

Clearly Theorem A follows from the following theorem:

Theorem 4.1. $\mathcal{D}(O_Q)^{\widetilde{G}_Q, \chi} = 0$

4.1. Proof of theorem 4.1. We denote by $\Gamma = \{w \in W : Q(w) = 1\}$. Note that by Witt's theorem Γ is an O_Q transitive set and therefore $\Gamma \times \Gamma$ is a transitive \widetilde{G}_Q set where the action of G_Q is the standard action on $W \oplus W$ and τ acts by flip.

Applying Frobenius reciprocity (3.3, 3.4) to projections of $O_Q \times \Gamma \times \Gamma$ first on $\Gamma \times \Gamma$ and then on O_Q we have

$$\mathcal{D}(O_Q)^{\widetilde{G}_Q, \chi} = \mathcal{D}(O_Q \times \Gamma \times \Gamma)^{\widetilde{G}_Q, \chi}$$

and also that

$$\mathcal{D}(O_Q \times \Gamma \times \Gamma)^{\widetilde{G}_Q, \chi} = \mathcal{D}(\Gamma \times \Gamma)^{\widetilde{H}_Q, \chi}$$

In what follows we will abuse notation and write $Q(u, v)$ for the bilinear form defined by Q . Define a map $D : \Gamma \times \Gamma \rightarrow Z$ where $Z = \{(v, u) \in W \oplus W : Q(v, u) = 0, Q(v + u) = 4\}$ by

$$D(x, y) = (x + y, x - y).$$

D defines an \tilde{G}_Q -equivariant homeomorphism and thus we need to show that

$$\mathcal{D}(Z)^{\widetilde{H_Q}, \chi} = 0$$

Here, the action of G_Q on $Z \subset W \oplus W$ is the restriction of its action on $W \oplus W$ while the action of τ is given by $\tau(v, u) = (v, -u)$.

Now we cover $Z = U_1 \cup U_2$ where

$$U_1 = \{(v, u) \in Z : Q(v) \neq 0\}$$

and

$$U_2 = \{(v, u) \in Z : Q(u) \neq 0\}$$

We will show $\mathcal{D}(U_1)^{\widetilde{H_Q}, \chi} = 0$, and the proof for U_2 is analogous. This will finish the proof.

Lemma 4.2. $\mathcal{D}(U_1)^{\widetilde{H_Q}, \chi} = 0$

Proof for non-archimedean F . Consider $\ell_1 : U_1 \rightarrow F - \{0\}$ defined as $\ell_1(v, u) = Q(v)$. By the localization principle, it is enough to show $\mathcal{D}(U_1^\alpha)^{\widetilde{H_Q}, \chi} = 0$ where $U_1^\alpha = \ell_1^{-1}(\alpha)$, for any $\alpha \in F - \{0\}$. But

$$U_1^\alpha = \{(v, u) | Q(v) = \alpha, Q(u) = 4 - \alpha, Q(v, u) = 0\}$$

Let $W^\alpha = \{w \in W | Q(w) = \alpha\}$ and let $p_1 : U_1^\alpha \rightarrow W^\alpha$ be given by $p_1(v, u) = v$.

On W^α our group acts transitively. Fix a vector $v_0 \in W^\alpha$.

Denote $H(v_0) := H_{(Q|_{v_0^\perp})}$ and $\tilde{H}(v_0) := \tilde{H}_{(Q|_{v_0^\perp})}$.

The stabilizer in \tilde{H}_Q of v_0 is $\tilde{H}(v_0)$. The fiber $p_1^{-1}(v_0) = \{a \in v_0^\perp | Q(a) = 4 - \alpha\}$. Frobenius reciprocity implies that

$$\mathcal{D}(U_1^\alpha)^{\widetilde{H_Q}, \chi} = \mathcal{D}(p_1^{-1}(v_0))^{\tilde{H}(v_0), \chi}$$

But clearly $\mathcal{D}(p_1^{-1}(v_0))^{\tilde{H}(v_0), \chi} = 0$ as $-Id \in H(v_0)$. \square

Proof for archimedean F . Now let us consider the archimedean case. Define $U := \{(v, u) \in U_1 | u \neq 0\}$. Note that the map $\ell_1|_U$ is a submersion, so the same argument as in the non-archimedean case shows that $\mathcal{D}(U)^{\widetilde{H_Q}, \chi} = 0$. Let $Y := \{(v \in W | Q(v) = 4\} \times \{0\}$ be the complement to U in U_1 . By theorem 3.2, it is enough to prove $\mathcal{D}(Y, \text{Sym}^k(CN_Y^{U_1}))^{\widetilde{H_Q}, \chi} = 0$.

Note that the action of \tilde{H}_Q on Y is transitive, and fix a point $(v, 0) \in Y$. The stabilizer in \tilde{H}_Q of $(v, 0)$ is $\tilde{H}(v)$, and the normal space to Y at $(v, 0)$ is v^\perp . So Frobenius reciprocity (theorem 3.4) implies that

$$\mathcal{D}(Y, \text{Sym}^k(CN_Y^{U_1}))^{\widetilde{H_Q}, \chi} = \text{Sym}^k(v^\perp)^{\tilde{H}(v), \chi}$$

But clearly $\text{Sym}^k(v^\perp)^{\tilde{H}(v), \chi} = 0$ as $-Id \in H(v)$. \square

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